
Z
X


Shutterstock.com/marekuliasz

2

TERMINOLOGY

argument
Cartesian form
De Moivre's theorem
imaginary part
modulus
polar form
real part
rectangular

COMPLEX NUMBERS

COMPLEX NUMBERS AND DE MOIVRE'S THEOREM

Prior learning

CARTESIAN FORMS

- review real and imaginary parts $Re(z)$ and $Im(z)$ of a complex number z (ACMSM077)
- review Cartesian form (ACMSM078)
- review complex arithmetic using Cartesian forms. (ACMSM079)

COMPLEX ARITHMETIC USING POLAR FORM

- use the modulus $|z|$ of a complex number z and the argument $Arg\ (z)$ of a non-zero complex number z and prove basic identities involving modulus and argument (ACMSM080)
- convert between Cartesian and polar form (ACMSM081)
- define and use multiplication, division, and powers of complex numbers in polar form and the geometric interpretation of these (ACMSM082)
- prove and use De Moivre's theorem for integral powers. (ACMSM083) AC

2.01 REVIEW OF COMPLEX NUMBERS

In this chapter you will review the concepts you studied in Year 11 relating to Complex Numbers and then develop further concepts. Recall the following definitions and rules.

IMPORTANT

The **imaginary number** $\boldsymbol{i}$ is the number such that $i = \sqrt{-1}$.

A **complex number** is a number that can be written in the form $a + ib$, where a and b are real numbers.

A complex number is often denoted by the letter z, so $z = a + ib$.

Example 1

Solve the quadratic equation $x^2 - 8x + 25 = 0$, expressing your answers in the form $a + bi$, where $a, b \in \boldsymbol{R}$.

Solution

Note that $x^2 - 8x + 25$ does not factorise, so it is necessary to solve by using the quadratic formula (or by completing the square).

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Substitute and simplify.

$$x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(25)}}{2(1)}$$

$$= \frac{8 \pm \sqrt{-36}}{2}$$

Recall that $\sqrt{-36} = \sqrt{36 \times (-1)} = 6 \times i$

$$\therefore x = \frac{8 \pm 6i}{2} = 4 \pm 3i$$

9780170250306

Alternate solution Complete the square.	$x^2 - 8x + 16 + 9 = 0$ $(x-4)^2 + 9 = 0$
Make a difference of two squares by first writing $+9 = -9i^2$.	$(x-4)^2 - 9i^2 = 0$ $(x-4-3i)(x-4+3i) = 0$
Isolate x.	$\therefore\ x = 4 \pm 3i$.

Recall the definition of the complex conjugate.

IMPORTANT

For a complex number z, where $z = a + ib$ (where a and b are real numbers), the **complex conjugate** is $\overline{z} = a - ib$.

Example 2

If $z = x + yi$, where $x, y \in \boldsymbol{R}$, prove that $z\overline{z}$ is always real.

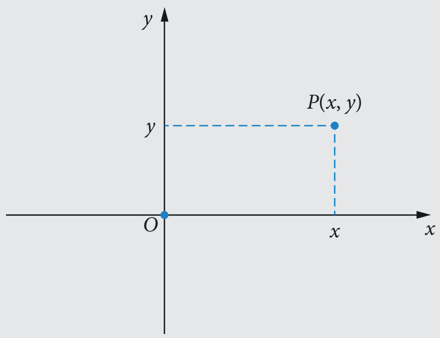
Solution

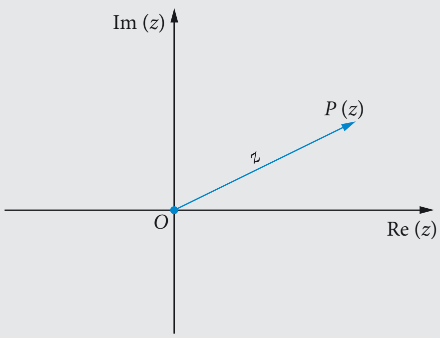
Note that $\overline{z} = x - yi$.	$z\overline{z} = (x + yi)(x - yi)$
Expand and simplify.	$z\overline{z} = x^2 - y^2i^2$ $= x^2 - [y^2 \times (-1)]$ $= x^2 + y^2$
There are no terms that include an i, so all terms are real.	Since $z\overline{z} = x^2 + y^2$, $z\overline{z}$ is always real.

Recall that it is possible to represent the complex number $z = x + yi$ (where $x, y \in \boldsymbol{R}$) geometrically on an **Argand diagram** or **Argand plane**.

IMPORTANT

The complex number $z = x + yi$ (where $x, y \in \boldsymbol{R}$) can be represented geometrically on an Argand diagram as the point $P(x, y)$ or the vector **z** or **OP**.





Example 3

For the complex number $z = 3 - 2i$,

a state the point P representing z on an Argand plane

b plot the vectors $\mathbf{z}$ and $\overline{\mathbf{z}}$ that represent z and its conjugate $\overline{z}$ on an Argand plane

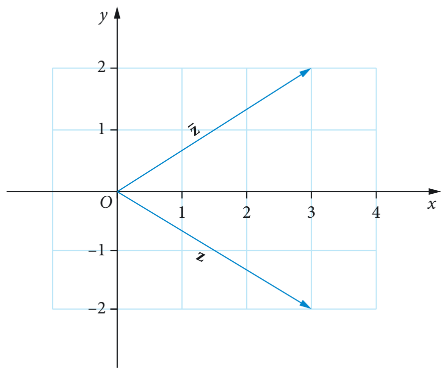
c describe the relationship between the vectors $\mathbf{z}$ and $\overline{\mathbf{z}}$.

Solution

a The point P representing z is an ordered pair.

$P(3, -2)$.

b The conjugate $\overline{z}$ is $3 + 2i$.



c The vectors $\mathbf{z}$ and $\overline{\mathbf{z}}$ have the same real part and opposite imaginary parts.

The conjugate vectors are reflections of each other over the real axis.

EXERCISE 2.01 Review of complex numbers

Concepts and techniques

1 Example 1 Use the quadratic formula to solve each equation, giving your solutions in the form $a + bi$, where $a, b \in \boldsymbol{R}$.

a $x^2 + 2x + 3 = 0$

b $x^2 - 4x + 7 = 0$

c $z^2 - 5iz - 6 = 0$

d $w^2 + w + 2 = 0$

2 Solve each equation below by completing the square. Check that your solutions are in the form $a \pm bi$, where $a, b \in \boldsymbol{R}$.

a $x^2 - 2x + 2 = 0$

b $y^2 + 4y + 5 = 0$

c $z^2 - 6z + 13 = 0$

d $w^2 + 8w + 18 = 0$.

3 Evaluate each of the following, giving your answer in simplest form.

a i^3

b i^4

c i^{50}

d $i + i^2 + \ldots + i^{11} + i^{12}$

9780170250306

4 Example 2 Consider the complex number $w = 3 - 5i$.

a State the conjugate $\overline{w}$.

b Find the value of $w \times \overline{w}$.

5 Write a quadratic equation in the form $az^2 + bz + c = 0$ with the following roots.

a $2 \pm i$ b $\sqrt{3} \pm 2i$ c $-1 \pm i\sqrt{5}$

6 Using your answers to question **2**, complete the sentence.

If a quadratic equation has real coefficients and one root is $p + qi$, where $p, q \in \boldsymbol{R}$, then the other root will be ….

7 Example 3 The vectors **u, v** and **w** represent the complex numbers $u = 1 - 2i$, $v = 2 + i$ and $w = -3 - 3i$. Sketch **u, v** and **w** on an Argand diagram.

Reasoning and communication

8 By considering the coefficients of the equation $x^2 + 2x + 5 = 0$, decide whether or not the roots are complex conjugates. Solve the equation to see if your prediction is true.

9 Show that the roots of $x^2 - 3ix - 3 + i = 0$ are $1 + i$ and $2i - 1$. Explain why the roots are not complex conjugates.

10 Evaluate $\sum_{r=0}^{r=197} i^r$.

11 Consider the point P representing the complex number $z = -2 + 3i$.

a Sketch the vector **OP** on an Argand plane.

b Sketch the vector **OQ** representing $w = \overline{-2 + 3i}$.

c Describe the geometric relationship between **OP** and **OQ**.

d Explain why **OP** – **OQ** lies on the y-axis.

2.02 REVIEW OF COMPLEX NUMBER OPERATIONS

You can perform all four operations with complex numbers: addition, subtraction, multiplication and division. This relies on grouping or equating the **real** and **imaginary parts**.

IMPORTANT

For complex numbers $a + ib$ and $c + id$ (where a, b, c and d are real numbers), $a + ib = c + id$ if and only if $a = c$ AND $b = d$.

Example 4

If $z = 2 + i$ and $w = -3 + 2i$, find

a $2z - w$ b w^2

Solution

a Substitute and group the real and imaginary parts.

$$\begin{aligned} 2z - w &= 2(2+i) - (-3+2i) \\ &= 4 + 2i + 3 - 2i \\ &= 7 \end{aligned}$$

b Substitute and expand using the identity $(a+b)^2 = a^2 + 2ab + b^2$.

$$\begin{aligned} w^2 &= (-3+2i)^2 \\ &= (-3)^2 + 2(-3)(2i) + (2i)^2 \end{aligned}$$

Simplify and recall that $i^2 = -1$.

$$\begin{aligned} &= 9 - 12i + 4i^2 \\ &= 9 - 12i - 4 \\ &= 5 - 12i \end{aligned}$$

Remember that you multiply by the complex conjugate to **realise the denominator**.

IMPORTANT

To **realise the denominator** of a complex number, multiply the number by 1 in the form $\frac{\overline{z}}{\overline{z}}$.

Example 5

Express $\frac{3+i}{3-i}$ in the form $a + bi$, where a, b are real.

Solution

Realise the denominator by multiplying by the conjugate.

$$\begin{aligned} \frac{3+i}{3-i} &= \frac{3+i}{3-i} \times \frac{3+i}{3+i} \\ &= \frac{9+6i+i^2}{9-i^2} \end{aligned}$$

Simplify and split into real and imaginary parts.

$$\begin{aligned} &= \frac{9+6i-1}{9-(-1)} \\ &= \frac{8+6i}{10} \\ &= \frac{8}{10} + \frac{6}{10}i \end{aligned}$$

Write the answer in its simplest form by cancelling any common factors.

$$\frac{3+i}{3-i} = \frac{4}{5} + \frac{3}{5}i$$

9780170250306

Recall the notation and language used for the real and imaginary parts of a complex number.

IMPORTANT

The **real part** of $z = a + ib$ is denoted by Re (z), where Re $(z) = a$ and the **imaginary part** by Im(z), where Im $(z) = b$.

If Re $(z) = 0$, then z is *purely imaginary.*

If Im $(z) = 0$, then z is *purely real* or just *real.*

Example 6

If $z = 5 + 7i - i(4 - 6i)$, find

a Re (z) b Im (z)

Solution

First expand and group into the real and imaginary parts.

$$\begin{aligned} z &= 5 + 7i - i(4 - 6i) \\ &= 5 + 7i - 4i + 6i^2 \\ &= 5 + 7i - 4i - 6 \\ &= -1 + 3i \end{aligned}$$

a If $z = a + bi$, where $a, b \in \boldsymbol{R}$, then Re $(z) = a$ — Re $(z) = -1$

b If $z = a + bi$, where $a, b \in \boldsymbol{R}$, then Im $(z) = b$ — Im $(z) = 3$

EXERCISE 2.02 Review of complex number operations

Concepts and techniques

WS Complex number operations

1 Example 4 Simplify each of the following expressions.

a $4(2 + i) + 3(2 - i)$ b $3(5 - 4i) - i(3 - 2i)$ c $(7 + 3i)^2$

d $\left(\sqrt{2}+3i\right)\left(\sqrt{2}-3i\right)$ e $(9 - i)(5 + 2i)$

2 Find the values of u and v, where u and v are real, if

a $u + vi = 2(3 - 4i) + i(5 + i)$ b $u + vi = -4(1 + 2i) - 3i(2 - 4i)$

3 If $z = 2 - i$ and $w = -4 + 5i$, find:

a zw b $\overline{z-w}$ c $\bar{z}\times w^2$ d $(z+i)(w-1)$

4 Example 5 Realise the denominator for each of the following complex fractions.

a $\dfrac{1}{2-i}$ b $\dfrac{1+2i}{3-2i}$ c $\dfrac{\sqrt{5}-i}{\sqrt{5}+i}$

5 Show that $\dfrac{1}{x-y+i(x+y)} = \dfrac{x-y-i(x+y)}{2\left(x^2+y^2\right)}$.

6 Show that $\frac{1+i\sqrt{3}}{1-i\sqrt{3}}+\frac{1-i\sqrt{3}}{1+i\sqrt{3}}$ is always real.

7 Example 6 Find Re (z) and Im (z) for each of the following.

a $z=-2\sqrt{3}-3i\sqrt{2}$

b $z=2(5+6i)+3(1-7i)$

c $z=x-yi-4w+vi$, where x, y, w and v are real.

d $z=\frac{3-i\sqrt{2}}{1+i\sqrt{2}}$

Reasoning and communication

8 Find Re (z) and Im (z) if $z=\frac{x-1+yi}{x-1-yi}$, where x, y are real.

9 Given $z=u+vi$, where u, v are real, find $\frac{z}{\bar{z}}$. Hence show that

$$\text{Re}\left(\frac{z}{\bar{z}}\right)=\frac{u^2-v^2}{u^2+v^2} \text{ and Im}\left(\frac{z}{\bar{z}}\right)=\frac{2uv}{u^2+v^2}$$

10 Simplify $\frac{1}{(x-yi)^2}+\frac{1}{(x+yi)^2}$.

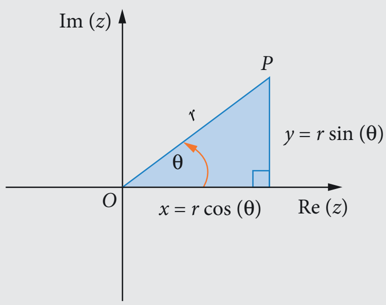
2.03 COMPLEX NUMBERS IN POLAR FORM

You know that a complex number z can be expressed in the form $x+yi$, where $x, y \in \boldsymbol{R}$.

This form is known as the **Cartesian form** (or **rectangular form**) as it uses x and y coordinates that can refer to axes. It is also convenient to express a complex number z in **polar form**, which uses the size of z and the angle that z makes with the positive x-axis.

IMPORTANT

The **argument** of the complex number $z \neq 0$, **arg** (z), is the angle θ that OP makes with the positive real axis, where P is the point that represents z in the complex plane. The **principal value** of the argument is the one in the interval $(-\pi, \pi]$. The argument of 0 is not defined.



The **modulus** of z is the magnitude of the vector $\mathbf{z}$, given by mod $(z)=|z|=\sqrt{x^2+y^2}$.

The **polar form** of z is given by $z=r[\cos(\theta)+i\sin(\theta)]$, $\cos(\theta)+i\sin(\theta)$ is often abbreviated to **cis** (θ).

The polar form is sometimes referred to as the **modulus-argument form**.

9780170250306

Example 7

Write down the complex numbers in polar form, given the following arguments and moduli.

a $\arg(z) = \frac{\pi}{3}$, $\mathrm{mod}(z) = 6$

b $\arg(w) = \frac{3\pi}{4}$, $|w| = 2$

Solution

a Recall that $\arg(z) = \theta$, $\mathrm{mod}(z) = r$. Substitute these into the formula.

$$z = r\left[\cos(\theta) + i\sin(\theta)\right]$$
$$= 6\left[\cos\left(\tfrac{\pi}{3}\right) + i\sin\left(\tfrac{\pi}{3}\right)\right]$$

b Recall that $|w| = r$. Substitute in the values.

$$w = r\left[\cos(\theta) + i\sin(\theta)\right]$$
$$= 2\left[\cos\left(\tfrac{3\pi}{4}\right) + i\sin\left(\tfrac{3\pi}{4}\right)\right]$$

IMPORTANT

The Cartesian and polar forms of the complex number z are related by the following equations.

$r = \sqrt{x^2 + y^2}$, $\tan(\theta) = \frac{y}{x}$, $x = r\cos(\theta)$ and $y = r\sin(\theta)$.

WS Modulus and argument

You can convert complex numbers between the Cartesian and polar forms using the relationships above.

Example 8

Convert the complex numbers to polar form.

a $z = 1 + i\sqrt{3}$ **b** $u = -2 - 2i$

Solution

a Find r.

$$|z| = r = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2$$

Find θ.

$$\cos(\theta) = \frac{1}{2} > 0,\ \sin(\theta) = \frac{\sqrt{3}}{2} > 0 \Rightarrow \theta = \frac{\pi}{3}$$

Write in polar form.

$$z = r\left[\cos(\theta) + i\sin(\theta)\right]$$
$$= 2\left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right]$$

b Find r.

$$|u| = r = \sqrt{(-2)^2 + (-2)^2} = 2\sqrt{2}$$

Find the principle value of θ.

$$\cos(\theta) = \frac{-2}{2\sqrt{2}} = \frac{-1}{\sqrt{2}} < 0$$

$$\sin(\theta) = \frac{-2}{2\sqrt{2}} = \frac{-1}{\sqrt{2}} < 0 \Rightarrow \theta = -\frac{3\pi}{4}$$

Write in polar form.

$$u = r\left[\cos(\theta) + i\sin(\theta)\right]$$
$$= 2\sqrt{2}\left[\cos\left(\frac{-3\pi}{4}\right) + i\sin\left(\frac{-3\pi}{4}\right)\right]$$

Example 9

Convert each complex number below to Cartesian form.

a $z=4\left[\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)\right]$ b $v=\sqrt{2}\left[\cos\left(\frac{\pi}{4}\right)-i\sin\left(\frac{\pi}{4}\right)\right]$

Solution

a Evaluate $\cos\left(\frac{2\pi}{3}\right)$ and $\sin\left(\frac{2\pi}{3}\right)$. $z=4\left[-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right]$

Expand. $=-2+2i\sqrt{3}$

b Evaluate $\cos\left(\frac{\pi}{4}\right)$ and $\sin\left(\frac{\pi}{4}\right)$. $v=\sqrt{2}\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i\right)$

Expand. $=1-i$

Note that a complex number in the form $z=r[\cos(\theta)-i\sin(\theta)]$ can be converted to polar form using trigonometric identities. You can write $z=r[\cos(\theta)-i\sin(\theta)]=r[\cos(-\theta)+i\sin(-\theta)]$ because $\cos(-\theta)=\cos(\theta)$ and $\sin(-\theta)=-\sin(\theta)$.

EXERCISE 2.03 Complex numbers in polar form

WS Complex number conversions

Concepts and techniques

1 Example 7 Find the complex number in the form $r[\cos(\theta)+i\sin(\theta)]$ for which

a $\arg(z)=\pi$, $|z|=5$ b $\arg(z)=\frac{\pi}{6}$, $\mathrm{mod}(z)=4$

c $\theta=\frac{-\pi}{2}$, $r=2$ d $\arg(v)=\frac{-\pi}{7}$, $\mathrm{mod}(v)=3\sqrt{2}$

2 For each complex number below, state its argument and modulus.

a $2\left[\cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)\right]$ b $2\sqrt{2}\left[\cos\left(\frac{5\pi}{6}\right)+i\sin\left(\frac{5\pi}{6}\right)\right]$

c $\cos\left(\frac{-\pi}{9}\right)+i\sin\left(\frac{-\pi}{9}\right)$ d $\frac{1}{\sqrt{3}}\left[\cos\left(\frac{-\pi}{3}\right)+i\sin\left(\frac{-\pi}{3}\right)\right]$

3 Example 8 For each complex number z below, calculate mod (z) and arg (z) in exact form.

a $z=\sqrt{3}+i$ b $z=3+3i$ c $z=\frac{1}{2}-\frac{1}{2}i$

d $z=-\frac{1}{2}+\frac{\sqrt{3}}{2}i$ e $z=7i$ f $z=-6$

4 Convert each complex number below to polar form.

a $z=\sqrt{2}+i\sqrt{2}$ b $w=-1+i\sqrt{3}$ c $u=\frac{\sqrt{3}}{2}-\frac{1}{2}i$

d $v=-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i$ e $z=-i\sqrt{5}$ f $w=1$

5 Example 9 Convert each complex number below to Cartesian form.

a $12\left[\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)\right]$ b $\sqrt{2}\left[\cos\left(\frac{3\pi}{4}\right)+i\sin\left(\frac{3\pi}{4}\right)\right]$

c $\cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right)$ d $\dfrac{\cos\left(\frac{-7\pi}{6}\right)+i\sin\left(\frac{-7\pi}{6}\right)}{2}$

6 Express each complex number below, with given arg and mod values, in Cartesian form.

a $\arg(z) = -\frac{\pi}{6}$, $|z| = 8$

b $\arg(z) = \frac{5\pi}{3}$, mod $(z) = 3$

c $\theta = \pi$, $r = 9$

d $\arg(v) = \frac{\pi}{3}$, $\text{mod}(v) = \sqrt{27}$

7 Simplify each complex number below, expressing your answer in $x + yi$ form.

a $2\cos\left(\frac{\pi}{6}\right) + 3i\sin\left(\frac{\pi}{3}\right)$

b $\sqrt{8}\left[\cos\left(\frac{-5\pi}{4}\right) + i\sin\left(\frac{\pi}{6}\right)\right]$

Reasoning and communication

8 Use trigonometric identities to convert each complex number to polar form.

a $r[\cos(\theta) - i\sin(\theta)]$

b $r[-\cos(\theta) + i\sin(\theta)]$

c $-r[\cos(\theta) + i\sin(\theta)]$

9 a Show that the points given by $P(r\cos(\theta), r\sin(\theta))$ for $0 \le \theta < 2\pi$ form a circle in the Cartesian plane.

b Show that the complex number $r\sin(\theta) + ri\cos(\theta)$ can be expressed in polar form.

c If $x + yi = 3\cos(\theta) + 4i\sin(\theta)$, show that $\frac{x^2}{9} + \frac{y^2}{16} = 1$.

2.04 MODULUS, ARGUMENT AND PRINCIPAL VALUE

$\frac{5\pi}{3}$ and $-\frac{\pi}{3}$ represent the same angle on the Cartesian or Argand plane. This means that a complex number can also be expressed in infinitely many ways. For example, $z = 2\left[\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right]$ is the same as $z = 2\left[\cos\left(\frac{-5\pi}{4}\right) + i\sin\left(\frac{-5\pi}{4}\right)\right]$. To avoid ambiguity, a complex number is usually expressed in polar form using the principal argument. In this case, $z = 2\left[\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right]$ rather than $z = 2\left[\cos\left(\frac{-5\pi}{4}\right) + i\sin\left(\frac{-5\pi}{4}\right)\right]$ is the convention since $-\pi < \frac{3\pi}{4} \le \pi$ and $\frac{3\pi}{4}$ is the principal argument.

Multiplying and dividing complex numbers in modulus-argument form is much easier than in the Cartesian form. Adding and subtracting complex numbers is easier using Cartesian form.

Example 10

Evaluate $z_1 \times z_2$ if $z_1 = 2\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right]$ and $z_2 = 5\left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right]$.

Solution

$$z_1 \times z_2 = 2\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right] \times 5\left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right]$$

Use the distributive law.

$$= 10\left[\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right) + i\cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right) + i\cos\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{4}\right) + i^2\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right)\right]$$

Rearrange into real and imaginary parts.

$$= 10\left\{\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right) - \sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right) + i\left[\cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{4}\right)\right]\right\}$$

Use the trigonometric identities.

$$= 10\left[\cos\left(\frac{\pi}{4} + \frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{4} + \frac{\pi}{6}\right)\right]$$

$$\therefore z_1 \times z_2 = 10\left[\cos\left(\frac{5\pi}{12}\right) + i\sin\left(\frac{5\pi}{12}\right)\right]$$

The example above demonstrates that when multiplying, you multiply two complex numbers in polar form by multiplying the moduli and adding the arguments.

IMPORTANT

If $z_1 = r_1[\cos(\theta_1) + i\sin(\theta_1)]$ and $z_2 = r_2[\cos(\theta_2) + i\sin(\theta_2)]$ are two complex numbers, then

- their **product** is $z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$
- $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$ and $\text{mod}(z_1 z_2) = \text{mod}(z_1) \times \text{mod}(z_2)$

Example 11

Find the product of $z_1 = 3\left[\cos\left(\frac{3\pi}{5}\right) + i\sin\left(\frac{3\pi}{5}\right)\right]$ and $z_2 = 2\left[\cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{4\pi}{5}\right)\right]$.

Solution

Use the rule $z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$

$$z_1 z_2 = 3\left[\cos\left(\frac{3\pi}{5}\right) + i\sin\left(\frac{3\pi}{5}\right)\right] \times 2\left[\cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{4\pi}{5}\right)\right]$$
$$= 3 \times 2\left[\cos\left(\frac{3\pi}{5} + \frac{4\pi}{5}\right) + i\sin\left(\frac{3\pi}{5} + \frac{4\pi}{5}\right)\right]$$

Simplify. Note that the principal value of the argument must be in the interval $(-\pi, \pi]$.

$$z_1 z_2 = 6\left[\cos\left(\frac{7\pi}{5}\right) + i\sin\left(\frac{7\pi}{5}\right)\right]$$
$$= 6\left[\cos\left(\frac{-3\pi}{5}\right) + i\sin\left(\frac{-3\pi}{5}\right)\right]$$

In a similar way, it is possible to develop rules for division of two complex numbers in polar form.

IMPORTANT

If $z_1 = r_1[\cos(\theta_1) + i\sin(\theta_1)]$ and $z_2 = r_2[\cos(\theta_2) + i\sin(\theta_2)]$ are two complex numbers, then

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right].$$

If $z = r[\cos(\theta) + i\sin(\theta)]$ is a complex number, then

$$z^{-1} = \frac{1}{z} = \frac{1}{r}\left[\cos(-\theta) + i\sin(-\theta)\right].$$

OR

If $z_1 = r_1[\cos(\theta_1) + i\sin(\theta_1)]$ and $z_2 = r_2[\cos(\theta_2) + i\sin(\theta_2)]$ are two complex numbers, then

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) \text{ and } \text{mod}\left(\frac{z_1}{z_2}\right) = \frac{\text{mod}(z_1)}{\text{mod}(z_2)}.$$

If $z = r[\cos(\theta) + i\sin(\theta)]$ is a complex number, then

$$\arg(z^{-1}) = -\arg(z) \text{ and } \text{mod}(z^{-1}) = \frac{1}{\text{mod}(z)}.$$

9780170250306

Example 12

If $z=\sqrt{2}\left[\cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)\right]$ and $w=2\sqrt{2}\left[\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)\right]$, find

a $\frac{z}{w}$ **b** z^{-1}

Solution

a Use the rule $\frac{z_1}{z_2}=\frac{r_1}{r_2}\left[\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2)\right]$, then simplify.

$$\frac{z}{w}=\frac{\sqrt{2}}{2\sqrt{2}}\left[\cos\left(\frac{\pi}{6}-\frac{2\pi}{3}\right)+i\sin\left(\frac{\pi}{6}-\frac{2\pi}{3}\right)\right]$$
$$=\frac{1}{2}\left[\cos\left(-\frac{3\pi}{6}\right)+i\sin\left(-\frac{3\pi}{6}\right)\right]$$
$$=\frac{1}{2}\left[\cos\left(-\frac{\pi}{2}\right)+i\sin\left(-\frac{\pi}{2}\right)\right]$$

b Use the rule $z^{-1}=\frac{1}{z}=\frac{1}{r}\left[\cos(-\theta)+i\sin(-\theta)\right]$

$$z^{-1}=\frac{1}{\sqrt{2}}\left[\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right]$$

EXERCISE 2.04 Modulus, argument and principal value

Concepts and techniques

1 Example 10 Expand, then use the identities $\sin(A+B)=\sin(A)\cos(B)+\cos(A)\sin(B)$ and $\cos(A+B)=\cos(A)\cos(B)-\sin(A)\sin(B)$ to simplify:

a $\left[\cos\left(\frac{\pi}{5}\right)+i\sin\left(\frac{\pi}{5}\right)\right]\left[\cos\left(\frac{\pi}{7}\right)+i\sin\left(\frac{\pi}{7}\right)\right]$

b $\left[\cos\left(\frac{2\pi}{9}\right)+i\sin\left(\frac{2\pi}{9}\right)\right]\left[\cos\left(\frac{\pi}{9}\right)+i\sin\left(\frac{\pi}{9}\right)\right]$

c $[\cos(\alpha)+i\sin(\alpha)][\cos(4\alpha)+i\sin(4\alpha)]$

d $[\cos(-3\beta)+i\sin(-3\beta)][\cos(-7\beta)+i\sin(-7\beta)]$

2 Express each of the following using a principal argument.

a $\text{cis}\left(\frac{3\pi}{2}\right)$ **b** $\text{cis}\left(\frac{5\pi}{4}\right)$ **c** $3\,\text{cis}\left(\frac{9\pi}{5}\right)$ **d** $2\,\text{cis}\left(\frac{-7\pi}{4}\right)$

3 Expand and then simplify, expressing your answer in polar form.

$\left[\cos\left(\frac{\pi}{5}\right)-i\sin\left(\frac{\pi}{5}\right)\right]\left[\cos\left(\frac{3\pi}{7}\right)-i\sin\left(\frac{3\pi}{7}\right)\right]$

4 Example 11 Use the fact that for two complex numbers, $z_1z_2=r_1\text{cis}(\theta_1)\times r_2\text{cis}(\theta_2)=r_1r_2\text{cis}(\theta_1+\theta_2)$ to simplify the following, leaving your answer in polar form.

a $\sqrt{2}\,\text{cis}\left(\frac{\pi}{3}\right)\times\sqrt{5}\,\text{cis}\left(\frac{\pi}{4}\right)$ **b** $2\,\text{cis}\left(\frac{5\pi}{7}\right)\times 4\,\text{cis}\left(\frac{3\pi}{7}\right)$

c $-3\,\text{cis}\left(\frac{\pi}{8}\right)\times\text{cis}\left(\frac{\pi}{4}\right)$ **d** $\sqrt{6}\,\text{cis}\left(\frac{-5\pi}{9}\right)\times\sqrt{2}\,\text{cis}\left(\frac{-8\pi}{9}\right)$

5 Realise the denominators of the following, expressing your answer in polar form.

a $\frac{1}{\cos\left(\frac{\pi}{3}\right)-i\sin\left(\frac{\pi}{3}\right)}$ **b** $\frac{1}{\sin\left(\frac{3\pi}{7}\right)+i\cos\left(\frac{3\pi}{7}\right)}$ **c** $\frac{1}{2\,\text{cis}\left(\frac{-\pi}{4}\right)}$

6 Example 12 Use the rules $\frac{z_1}{z_2}=\frac{r_1\text{cis}(\theta_1)}{r_2\text{cis}(\theta_2)}=\frac{r_1}{r_2}\text{cis}(\theta_1-\theta_2)$ and $z^{-1}=\frac{1}{z}=\frac{1}{r\,\text{cis}(\theta)}=\frac{1}{r}\text{cis}(-\theta)$ to simplify:

a $\frac{6\text{ cis}\left(\frac{\pi}{2}\right)}{2\text{ cis}\left(\frac{\pi}{3}\right)}$

b $\frac{12\text{ cis}\left(\frac{-\pi}{3}\right)}{3\text{ cis}\left(\frac{2\pi}{3}\right)}$

c $\frac{15\text{ cis}\left(\frac{-\pi}{5}\right)}{5\text{ cis}\left(\frac{-3\pi}{5}\right)}$

d $\frac{1}{\text{cis}(\theta)}$

e $\frac{1}{3\text{ cis}\left(\frac{3\pi}{2}\right)}$

f $\frac{4}{\text{cis}\left(\frac{-5\pi}{4}\right)}$

7 If $z=\sqrt{3}\text{cis}\left(\frac{\pi}{3}\right)$ and $w=\text{cis}\left(\frac{3\pi}{4}\right)$, find:

a zw

b $\frac{z}{w}$

c $\frac{1}{zw}$

d z^2

e $(zw)^{-2}$

f $\overline{zw}$

8 Simplify each of the following, leaving your answer in polar form.

a $\left[\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)\right]\left[\cos\left(\frac{5\pi}{3}\right)-i\sin\left(\frac{5\pi}{3}\right)\right]$

b $[\cos(4)-i\sin(4)][\cos(-2)-i\sin(-2)]$

c $3\left[\cos\left(\frac{\pi}{2}\right)-i\sin\left(\frac{\pi}{2}\right)\right]\left[4\cos\left(\frac{3\pi}{4}\right)+4i\sin\left(\frac{3\pi}{4}\right)\right]$

Reasoning and communication

9 If $z=\cos(\theta)+i\sin(\theta)$, prove that $z^{-1}=\bar{z}$.

10 If $z=r[\cos(\theta)+i\sin(\theta)]$, prove that $z^{-1}=\frac{\bar{z}}{r^2}$.

11 If $z=r[\cos(\theta)+i\sin(\theta)]$ and $w=r[\cos(\alpha)+i\sin(\alpha)]$, show that $\text{Re}(zw)=r^2\cos(\theta+\alpha)$ and $\text{Im}(zw)=r^2\sin(\theta+\alpha)$.

2.05 OPERATIONS IN POLAR FORM

You can combine rules to solve problems.

Example 13

By first expressing $\sqrt{3}+i$ in polar form, find:

a $\arg\left(\sqrt{3}+i\right)^{-1}$

b $\left|\left(\sqrt{3}+i\right)^{-1}\right|$

Solution

Find the modulus, r, and argument, θ, of $\sqrt{3}+i$.

$$r=\left|\sqrt{3}+i\right|=\sqrt{\left(\sqrt{3}\right)^2+1^2}=2$$

$$\cos(\theta)=\frac{\sqrt{3}}{2},\ \sin(\theta)=\frac{1}{2}\Rightarrow\theta=\frac{\pi}{6}$$

$$\therefore\sqrt{3}+i=2\left[\cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)\right].$$

9780170250306

a Use the rule $\arg(z^{-1}) = -\arg(z)$.

$$\arg\left(\sqrt{3}+i\right)^{-1} = -\arg\left(\sqrt{3}+i\right) = -\frac{\pi}{6}$$

b Use the rule $\operatorname{mod}\left(z^{-1}\right) = \dfrac{1}{\operatorname{mod}(z)}$.

$$\left|\left(\sqrt{3}+i\right)^{-1}\right| = \frac{1}{\operatorname{mod}\left(\sqrt{3}+i\right)} = \frac{1}{2}$$

You can extend rules to multiple complex numbers.

IMPORTANT

If $z_1 = r_1[\cos(\theta_1) + i\sin(\theta_1)]$, $z_2 = r_2[\cos(\theta_2) + i\sin(\theta_2)]$, …, $z_n = r_n[\cos(\theta_n) + i\sin(\theta_n)]$ are multiple complex numbers, then

$$z_1 z_2 \ldots z_n = r_1 r_2 \ldots r_n[\cos(\theta_1 + \theta_2 + \ldots + \theta_n) + i\sin(\theta_1 + \theta_2 + \ldots + \theta_n)]$$

OR

$$\arg(z_1 z_2 \ldots z_n) = \arg(z_1) + \arg(z_2) + \ldots + \arg(z_n) \text{ and } |z_1 z_2 \ldots z_n| = |z_1||z_2|\ldots|z_n|.$$

If $z_1 = z_2 = \ldots = z_n$, then the following rules also hold.

IMPORTANT

$$\arg(z^n) = n \times \arg(z) \text{ and } |z^n| = |z|^n$$

and

$$\arg(z^{-n}) = -n \times \arg(z) \text{ and } |z^{-n}| = |z|^{-n}.$$

Example 14

If $z = \sqrt{2}\operatorname{cis}\left(\frac{3\pi}{4}\right)$, $w = \sqrt{3}\operatorname{cis}\left(\frac{\pi}{4}\right)$ and $u = \sqrt{2}\operatorname{cis}\left(\frac{\pi}{2}\right)$, find

a $\arg(zwu)$ b $\left|\dfrac{z}{w^2 u}\right|$

Solution

a Use the rule $\arg(z_1 z_2 \ldots z_n) = \arg(z_1) + \arg(z_2) + \ldots + \arg(z_n)$

$$\arg(zwu) = \frac{3\pi}{4} + \frac{\pi}{4} + \frac{\pi}{2} = \frac{3\pi}{2}$$

Express $\dfrac{3\pi}{2}$ in the domain $(-\pi, \pi]$.

$$\therefore \arg(zwu) = \frac{-\pi}{2}.$$

b Use $|z_1 z_2 \ldots z_n| = |z_1||z_2| \ldots |z_n|$ and $|z^n| = |z|^n$

$$\left|\frac{z}{w^2 u}\right| = \frac{|z|}{|w^2||u|} = \frac{|z|}{|w|^2|u|}$$

Now substitute and simplify.

$$\left|\frac{z}{w^2 u}\right| = \frac{\sqrt{2}}{\left(\sqrt{3}\right)^2\sqrt{2}} = \frac{1}{3}$$

A number of difficult calculations can be done by expressing complex numbers in polar form.

Example 15

Evaluate $\dfrac{1}{(1+i)\left(1-i\sqrt{3}\right)}$, leaving your answer in polar form.

Solution

First convert $1+i$ and $1-i\sqrt{3}$ to polar form.

$1+i=\sqrt{2}\left[\cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)\right]$ and

$1-i\sqrt{3}=2\left[\cos\left(\frac{-\pi}{3}\right)+i\sin\left(\frac{-\pi}{3}\right)\right]$.

Now use the rules $\arg(zw)=\arg(z)+\arg(w)$ and $\arg(z^{-1})=-\arg(z)$

$$\begin{aligned}\arg\left[\frac{1}{(1+i)\left(1-i\sqrt{3}\right)}\right]&=-\arg\left[(1+i)\left(1-i\sqrt{3}\right)\right]\\&=-\left[\arg(1+i)+\arg\left(1-i\sqrt{3}\right)\right]\\&=-\left(\frac{\pi}{4}-\frac{\pi}{3}\right)\\&=\frac{\pi}{12}\end{aligned}$$

Use the rules $|zw|=|z||w|$ and $|z^{-n}|=|z|^{-n}$.

$$\begin{aligned}\left|(1+i)\left(1-i\sqrt{3}\right)\right|&=\left|(1+i)\right|\left|\left(1-i\sqrt{3}\right)\right|\\&=\sqrt{2}\times 2\\&=2\sqrt{2}\end{aligned}$$

$$\therefore\left|\frac{1}{(1+i)\left(1-i\sqrt{3}\right)}\right|=\frac{1}{\left|(1+i)\left(1-i\sqrt{3}\right)\right|}=\frac{1}{2\sqrt{2}}$$

Express your answer in the form $r[\cos(\theta)+i\sin(\theta)]$.

$$\therefore\frac{1}{(1+i)\left(1-i\sqrt{3}\right)}=\frac{1}{2\sqrt{2}}\left[\cos\left(\tfrac{\pi}{12}\right)+i\sin\left(\tfrac{\pi}{12}\right)\right].$$

EXERCISE 2.05 Operations in polar form

Concepts and techniques

Constructing spirals

Polar complex number operations

1 Example 13 Convert each number to polar form and hence find $\arg(z)$ and $|z|$.

a $z=1-i$ **b** $z=\sqrt{3}+i$ **c** $z=-\sqrt{2}+i\sqrt{2}$ **d** $z=-\frac{1}{2}-\frac{\sqrt{3}}{2}i$

2 Find the argument of each complex number in radians, correct to 3 significant figures.

a $z=5-2i$ **b** $z=-2+3i$ **c** $z=-\sqrt{3}-2i$ **d** $z=1-i\sqrt{5}$

3 Evaluate each of the following.

a $\arg\left(\frac{1}{2}-\frac{i}{2}\right)$ b $|3[\cos(2)+i\sin(2)]|$ c $\arg(\text{cis}(\theta)\times\text{cis}(\alpha))$ d $\left|\frac{5\,\text{cis}\left(\frac{\pi}{9}\right)}{\sqrt{5}\,\text{cis}\left(\frac{\pi}{3}\right)}\right|$

4 Example 14 Use the theorems $\arg(z_1z_2\ldots z_n)=\arg(z_1)+\arg(z_2)+\ldots+\arg(z_n)$ and $|z_1z_2\ldots z_n|=|z_1||z_2|\ldots|z_n|$ to simplify:

a $[\cos(2\alpha)+i\sin(2\alpha)][\cos(3\alpha)+i\sin(3\alpha)][\cos(6\alpha)+i\sin(6\alpha)]$

b $2\left[\cos\left(\frac{\pi}{9}\right)+i\sin\left(\frac{\pi}{9}\right)\right]\times\sqrt{2}\left[\cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)\right]\times\sqrt{6}\left[\cos\left(\frac{-\pi}{18}\right)+i\sin\left(\frac{-\pi}{18}\right)\right]$

c $\left[\cos\left(\frac{3\pi}{7}\right)+i\sin\left(\frac{3\pi}{7}\right)\right]\left[3\cos\left(\frac{\pi}{7}\right)-3i\sin\left(\frac{\pi}{7}\right)\right]\left[\sqrt{5}\cos\left(\frac{-2\pi}{7}\right)-i\sqrt{5}\sin\left(\frac{-2\pi}{7}\right)\right]$

d $\dfrac{1}{\left[4\cos\left(\frac{5\pi}{11}\right)+4i\sin\left(\frac{5\pi}{11}\right)\right]^2\times3\left[\cos\left(\frac{3\pi}{5}\right)+i\sin\left(\frac{3\pi}{5}\right)\right]}$

5 Use the rules $|z^n|=|z|^n$ and $|z^{-n}|=|z|^{-n}$ to find

a $|\cos(\theta)+i\sin(\theta)|^5$ b $\left|3\left[\cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)\right]\right|^4$ c $\left|\frac{1}{\sqrt{7}[\cos(\theta)+i\sin(\theta)]}\right|^3$

6 Use the rules $\arg(z^n)=n\times\arg(z)$ and $\arg(z^{-n})=-n\times\arg(z)$ to simplify

a $\arg\{[\cos(\theta)+i\sin(\theta)]^6\}$ b $\arg\left\{\left[\cos\left(\frac{3\pi}{8}\right)+i\sin\left(\frac{3\pi}{8}\right)\right]^9\right\}$ c $\arg\left\{\frac{1}{\left[\cos\left(\frac{4\pi}{9}\right)+i\sin\left(\frac{4\pi}{9}\right)\right]^4}\right\}$

7 Example 15 First express each of the following in polar form and then simplify. Leave your answer in polar form where appropriate.

a $\left(\sqrt{3}-i\right)(1+i)$

b $\dfrac{\sqrt{2}+i\sqrt{2}}{1+i\sqrt{3}}$

c $\dfrac{1}{6\left(-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)}$

d $\dfrac{\left(2-2i\sqrt{3}\right)\left(-\sqrt{3}+i\right)}{(4+4i)}$

Reasoning and communication

8 a Express $(1+i)\left(1+i\sqrt{3}\right)$ in mod-arg form.

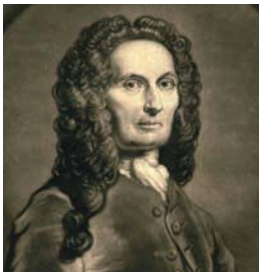
b Hence find the exact value of $\cos\left(\frac{7\pi}{2}\right)$.

9 If $r\left[\cos(\alpha)+i\sin(\alpha)\right]=\dfrac{3-i}{2+5i}$, find the exact value of $\sin(\alpha)$.

10 If $\{r[\cos(\alpha)+i\sin(\alpha)]\}^n=x+yi$, where x and y are real, prove that $r=\left(x^2+y^2\right)^{\frac{1}{2n}}$.

2.06 DE MOIVRE'S THEOREM

The rules in the previous section can be extended to powers of $z = r[\cos(\theta) + i\sin(\theta)]$. This work was developed by the French mathematician Abraham De Moivre (1667 – 1754) and is called De Moivre's theorem.



IMPORTANT

De Moivre's theorem

If $z = \cos(\theta) + i\sin(\theta)$ is a complex number, then $z^n = \cos(n\theta) + i\sin(n\theta)$, $\forall\, n \in \mathbf{Z}$.

Proof

The proof relies on mathematical induction.

Proposition

Let $P(n)$ be the proposition that $[\cos(\theta) + i\sin(\theta)]^n = \cos(n\theta) + i\sin(n\theta)$, $\forall\, n \in \mathbf{N}, n \geq 1$.

RTP

Both $P(1)$ is true and $P(k+1)$ is true given that $P(k)$ is true.

Proof

When $n = 1$,

$$\begin{aligned}\text{LHS} &= [\cos(\theta) + i\sin(\theta)]^1 \\ &= \cos(\theta) + i\sin(\theta)\end{aligned}$$

$$\begin{aligned}\text{RHS} &= [\cos(1 \times \theta) + i\sin(1 \times \theta)] \\ &= \cos(\theta) + i\sin(\theta)\end{aligned}$$

$\therefore$ LHS = RHS $\Rightarrow P(1)$ is true.

Assume that $P(k)$ is true, i.e. that

$[\cos(\theta) + i\sin(\theta)]^k = [\cos(k\theta) + i\sin(k\theta)]$, for some $k \in \mathbf{N}, k \geq 1$.

$P(k+1)$: $[\cos(\theta) + i\sin(\theta)]^{k+1} = \cos[(k+1)\theta] + i\sin[(k+1)\theta]$, for some $k \in \mathbf{N}, k \geq 1$.

Consider the LHS of $P(k+1)$:

$$\begin{aligned}[\cos(\theta) + i\sin(\theta)]^{k+1} &= [\cos(\theta) + i\sin(\theta)][\cos(\theta) + i\sin(\theta)]^k \\ &= [\cos(\theta) + i\sin(\theta)][\cos(k\theta) + i\sin(k\theta)], \text{ using } P(k) \\ &= \cos(\theta)\cos(k\theta) + \cos(\theta)\, i\sin(k\theta) + i\sin(\theta)\cos(k\theta) + i^2\sin(\theta)\sin(k\theta) \\ &= \cos(\theta)\cos(k\theta) - \sin(\theta)\sin(k\theta) + i[\cos(\theta)\sin(k\theta) + \sin(\theta)\cos(k\theta)] \\ &= \cos(\theta + k\theta) + i\sin(\theta + k\theta) \\ &= \cos[(k+1)\theta] + i\sin[(k+1)\theta] \\ &= \text{RHS of } P(k+1)\end{aligned}$$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true by mathematical induction. **QED**

9780170250306

Example 16

Use De Moivre's theorem to simplify the following.

a $[\cos(\theta) + i\sin(\theta)]^6$ **b** $[\cos(3\alpha) + i\sin(3\alpha)]^{-8}$ **c** $\left[\cos\left(\frac{\pi}{4}\right) - i\sin\left(\frac{\pi}{4}\right)\right]^4$

Solution

a Apply the theorem directly using the formula $[\cos(\theta) + i\sin(\theta)]^n = \cos(n\theta) + i\sin(n\theta)$

$$[\cos(\theta) + i\sin(\theta)]^6 = \cos(6\theta) + i\sin(6\theta)$$

b Let $\theta = 3\alpha$ and apply the theorem.

$$[\cos(3\alpha) + i\sin(3\alpha)]^{-8} = \cos(-24\alpha) + i\sin(-24\alpha)$$

c First, write the expression in polar form.

$$\left[\cos\left(\frac{\pi}{4}\right) - i\sin\left(\frac{\pi}{4}\right)\right]^4 = \left[\cos\left(\frac{-\pi}{4}\right) + i\sin\left(\frac{-\pi}{4}\right)\right]^4$$

Now apply the theorem and simplify.

$$\left[\cos\left(\frac{-\pi}{4}\right) + i\sin\left(\frac{-\pi}{4}\right)\right]^4 = \cos(-\pi) + i\sin(-\pi) = -1$$

De Moivre's theorem is useful for evaluating powers of complex numbers. First, convert the expressions to polar form.

Example 17

Convert $\frac{-1+i\sqrt{3}}{2}$ to polar form and hence evaluate $\left(\frac{-1+i\sqrt{3}}{2}\right)^8$, giving your answer in the form $a + bi$.

Solution

First, find r and θ.

$$r = \sqrt{\left(\frac{-1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1$$

$$\cos(\theta) = \frac{-1}{2},\ \sin(\theta) = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{2\pi}{3}$$

$$\therefore \frac{-1+i\sqrt{3}}{2} = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)$$

Now apply De Moivre's theorem and simplify.

$$\left[\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right]^8 = \cos\left(\frac{16\pi}{3}\right) + i\sin\left(\frac{16\pi}{3}\right) = \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)$$

$$\text{or } \cos\left(\frac{-2\pi}{3}\right) + i\sin\left(\frac{-2\pi}{3}\right)$$

Convert to Cartesian form.

$$\cos\left(\frac{-2\pi}{3}\right) + i\sin\left(\frac{-2\pi}{3}\right) = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

$$\therefore \left(\frac{-1+i\sqrt{3}}{2}\right)^8 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i.$$

You can now use a combination of rules to simplify expressions.

Example 18

Simplify $\dfrac{[\cos(2\beta)+i\sin(2\beta)]^7 \times [\cos(3\beta)+i\sin(3\beta)]^4}{[\cos(\beta)+i\sin(\beta)]^2}$

Solution

Use De Moivre's theorem to simplify each bracket.

$$\frac{[\cos(2\beta)+i\sin(2\beta)]^7 \times [\cos(3\beta)+i\sin(3\beta)]^4}{[\cos(\beta)+i\sin(\beta)]^2}$$

$$=\frac{[\cos(14\beta)+i\sin(14\beta)]\times[\cos(12\beta)+i\sin(12\beta)]}{\cos(2\beta)+i\sin(2\beta)}$$

Use $z_1z_2 = r_1r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$ and $\dfrac{z_1}{z_2}=\dfrac{r_1}{r_2}[\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2)]$

$= \cos(14\beta + 12\beta - 2\beta) + i\sin(14\beta + 12\beta - 2\beta)$

$= \cos(24\beta) + i\sin(24\beta)$

EXERCISE 2.06 De Moivre's theorem

Concepts and techniques

WS Using De Moivre's theorem

1 Example 16 Use De Moivre's theorem to simplify each complex number.

a $[\cos(\theta) + i\sin(\theta)]^9$

b $[\cos(\theta) + i\sin(\theta)]^{-2}$

c $[\cos(\theta) - i\sin(\theta)]^4$

d $[\cos(2\theta) + i\sin(2\theta)]^3$

2 Evaluate z^4 in Cartesian form if

a $z=2\left[\cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)\right]$

b $z=3\left[\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)\right]$

c $z=\frac{1}{\sqrt{2}}\left[\cos\left(\frac{\pi}{8}\right)+i\sin\left(\frac{\pi}{8}\right)\right]$

d $z=5\left[\cos\left(\frac{7\pi}{12}\right)+i\sin\left(\frac{7\pi}{12}\right)\right]$

3 Evaluate $\left[\cos\left(\frac{-3\pi}{5}\right)+i\sin\left(\frac{-3\pi}{5}\right)\right]^{-6}$.

4 Example 17 By first converting to polar form, evaluate each of the following, giving your answer in Cartesian form.

a $(1+i)^3$

b $\left(\sqrt{3}+i\right)^4$

c $\left(\sqrt{2}-i\sqrt{2}\right)^7$

d $\left(\dfrac{-1-i}{\sqrt{2}}\right)^{-6}$

5 Evaluate the following.

a $\dfrac{1}{(2+2i)^6}$

b $\dfrac{1}{\left(1-i\sqrt{3}\right)^8}$

6 Show that $\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)^{12}\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^3=1$

9780170250306

7 Example 18 Simplify each complex number.

a $[\cos(\theta) + i\sin(\theta)]^5 \times [\cos(\theta) + i\sin(\theta)]^{-8}$

b $[\cos(\alpha) + i\sin(\alpha)]^3 \times [\cos(\beta) + i\sin(\beta)]^4$

c $\dfrac{[\cos(4\phi) + i\sin(4\phi)]^5}{[\cos(7\phi) + i\sin(7\phi)]^3}$

d $\dfrac{[\cos(\beta) - i\sin(\beta)]^2 \times [\cos(\beta) + i\sin(\beta)]^{-3}}{[\cos(3\beta) + i\sin(3\beta)]^4}$

Reasoning and communication

8 If $z = \cos(\theta) + i\sin(\theta)$, prove that $z^2 + \dfrac{1}{z^2}$ is always real.

9 If $z = \cos(\theta) + i\sin(\theta)$, prove that $z^3 - \dfrac{1}{z^3}$ is purely imaginary.

10 If $z = \cos(\theta) + i\sin(\theta)$, what is the value of $z^n + \dfrac{1}{z^n}$?

2.07 APPLICATIONS OF DE MOIVRE'S THEOREM

De Moivre's theorem is useful for deriving or proving a number of trigonometric identities. These will be examined in this section.

Example 19

Use the binomial theorem to expand $[\cos(\theta) + i\sin(\theta)]^3$. Hence derive a formula for $\cos(3\theta)$ in terms of $\cos(\theta)$.

Solution

Recall the expansion for $(a + b)^3$.

$$[\cos(\theta) + i\sin(\theta)]^3$$
$$= \cos^3(\theta) + 3\cos^2(\theta)\,[i\sin(\theta)] + 3\cos(\theta)\,[i^2\sin^2(\theta)] + i^3\sin^3(\theta)$$
$$= \cos^3(\theta) + 3i\cos^2(\theta)\sin(\theta) - 3\cos(\theta)\sin^2(\theta) - i\sin^3(\theta)$$

Use De Moivre's theorem to find an expression with $\cos(3\theta)$.

$$[\cos(\theta) + i\sin(\theta)]^3 = \cos(3\theta) + i\sin(3\theta)$$

$$\therefore \cos(3\theta) + i\sin(3\theta)$$
$$= \cos^3(\theta) + 3i\cos^2(\theta)\sin(\theta) - 3\cos(\theta)\sin^2(\theta) - i\sin^3(\theta)$$

Equate the real parts.

$$\therefore \cos(3\theta) = \cos^3(\theta) - 3\cos(\theta)\sin^2(\theta)$$

Use a substitution to eliminate $\sin^2(\theta)$.

$$\cos(3\theta) = \cos^3(\theta) - 3\cos(\theta)[1 - \cos^2(\theta)]$$
$$= \cos^3(\theta) - 3\cos(\theta) + 3\cos^3(\theta)$$
$$= 4\cos^3(\theta) - 3\cos(\theta)$$
$$\therefore \cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta)$$

Example 20

By expanding $[\cos(\alpha) + i\sin(\alpha)]^2$ in two ways, derive expressions for $\cos(2\alpha)$ and $\sin(2\alpha)$ and hence find an expression for $\tan(2\alpha)$ in terms of $\tan(\alpha)$.

Solution

Use the binomial theorem.

$$[\cos(\alpha) + i\sin(\alpha)]^2 = \cos^2(\alpha) + 2i\cos(\alpha)\sin(\alpha) + i^2\sin^2(\alpha)$$
$$= \cos^2(\alpha) - \sin^2(\alpha) + 2i\cos(\alpha)\sin(\alpha)$$

Use De Moivre's theorem.

$$[\cos(\alpha) + i\sin(\alpha)]^2 = \cos(2\alpha) + i\sin(2\alpha)$$

Equate the real and imaginary parts.

$$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha)$$
$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$$

Use $\tan(A) = \frac{\sin(A)}{\cos(A)}$

$$\tan(2\alpha) = \frac{\sin(2\alpha)}{\cos(2\alpha)}$$
$$= \frac{2\sin(\alpha)\cos(\alpha)}{\cos^2(\alpha) - \sin^2(\alpha)}$$

Convert to $\tan(\alpha)$ by dividing the RHS by $\frac{\cos^2(\alpha)}{\cos^2(\alpha)}$.

$$\tan(2\alpha) = \frac{\frac{2\sin(\alpha)\cos(\alpha)}{\cos^2(\alpha)}}{\frac{\cos^2(\alpha)}{\cos^2(\alpha)} - \frac{\sin^2(\alpha)}{\cos^2(\alpha)}}$$
$$= \frac{2\tan(\alpha)}{1 - \tan^2(\alpha)}$$

Expansions using De Moivre's theorem and the binomial theorem can also be used to integrate powers of trigonometric functions.

Example 21

Use the fact that $\cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta)$ to find $\int_0^{\frac{\pi}{2}} 4\cos^3(\theta)\,d\theta$.

Solution

Rearrange $\cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta)$ to make $4\cos^3(\theta)$ the subject.

$$\cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta)$$
$$\therefore\ 4\cos^3(\theta) = \cos(3\theta) + 3\cos(\theta)$$

Perform the integration.

$$\int_0^{\frac{\pi}{2}} 4\cos^3(\theta)\,d\theta = \int_0^{\frac{\pi}{2}} \cos(3\theta) + 3\cos(\theta)\,d\theta$$
$$= \left[\frac{\sin(3\theta)}{3} + 3\sin(\theta)\right]_0^{\frac{\pi}{2}}$$
$$= \frac{\sin\left[3\left(\frac{\pi}{2}\right)\right]}{3} + 3\sin\left(\frac{\pi}{2}\right) - \left(\frac{\sin[3(0)]}{3} + 3\sin(0)\right)$$
$$= -\frac{1}{3} + 3(1) - (0)$$
$$= 2\frac{2}{3}$$

9780170250306

INVESTIGATION Euler and $e^{i\theta}$

Leonhard Euler was born in Switzerland in 1707. His father wanted him to study Theology, but Euler persuaded his father to allow him to study Mathematics. Euler was the first mathematician to use the notation $f(x)$ for a function and in 1748 he defined the formula $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.

Getty Images/Heritage Images

This definition was very useful in proving the various results to do with complex numbers, including De Moivre's theorem.

By expressing the complex number $z = r[\cos(\theta) + i\sin(\theta)]$ as $z = re^{i\theta}$, prove the following results.

1. $z_1 z_2 \ldots z_n = r_1 r_2 \ldots r_n[\cos(\theta_1 + \theta_2 + \ldots + \theta_n) + i\sin(\theta_1 + \theta_2 + \ldots + \theta_n)]$
2. $z^n = r^n[\cos(n\theta) + i\sin(n\theta)], \forall n \in \mathbf{Z}$
3. $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]$
4. $z^{-1} = \dfrac{1}{z} = \dfrac{1}{r}\left[\cos(-\theta) + i\sin(-\theta)\right]$

EXERCISE 2.07 Applications of De Moivre's theorem

Concepts and techniques

Trigonometric identities using De Moivre's theorem

1 Example 19 **a** Expand $[\cos(\theta) + i\sin(\theta)]^3$ using the binomial theorem.
   b Use the fact that $[\cos(\theta) + i\sin(\theta)]^3 = \cos(3\theta) + i\sin(3\theta)$ to derive an expression for
      i $\cos(3\theta)$ in terms of $\cos(\theta)$
      ii $\sin(3\theta)$ in terms of $\sin(\theta)$

2 **a** Expand $[\cos(\theta) + i\sin(\theta)]^2$ in two ways.
   b Hence show that
      i $\cos(2\theta) = 2\cos^2(\theta) - 1$
      ii $\cos(2\theta) = 1 - 2\sin^2(\theta)$

3 Use expansions of $[\cos(\theta) + i\sin(\theta)]^4$ to prove that
   a $\cos(4\theta) = 8\cos^4(\theta) - 8\cos^2(\theta) + 1$
   b $\sin(4\theta) = 4\sin(\theta)\cos(\theta)[\cos^2(\theta) - \sin^2(\theta)]$

4 Example 20 a Complete the statement $\tan(3\theta) = \dfrac{\_\_\_\_(3\theta)}{\cos(\_\_\_\_)}$.

b Using the results you derived from $[\cos(\theta) + i\sin(\theta)]^3 = \cos(3\theta) + i\sin(3\theta)$ in question **1**, show that $\tan(3\theta) = \dfrac{3\sin(\theta) - 4\sin^3(\theta)}{4\cos^3(\theta) - 3\cos(\theta)}$.

c Hence prove that $\tan(3\theta) = \dfrac{3\tan(\theta) - \tan^3(\theta)}{1 - 3\tan^2(\theta)}$.

5 For $n \geq 1$, prove that

$[\cos(\theta) + i\sin(\theta)] + [\cos(2\theta) + i\sin(2\theta)] + [\cos(3\theta) + i\sin(3\theta)] + \ldots + [\cos(n\theta) + i\sin(n\theta)]$

$= \dfrac{[\cos(\theta) + i\sin(\theta)][\cos(n\theta) + i\sin(n\theta) - 1]}{\cos(\theta) + i\sin(\theta) - 1}$

6 If $60° - 45° = 15°$, express cis (60°) and cis (45°) in Cartesian form and hence simplify $\dfrac{\text{cis}(60°)}{\text{cis}(45°)}$ in two ways. Use your results to find the exact value of cos (15°).

7 Example 21 a Use $[\cos(\theta) + i\sin(\theta)]^3 = \cos(3\theta) + i\sin(3\theta)$ to show that $\sin(3\alpha) = 3\sin(\alpha) - 4\sin^3(\alpha)$.

b Hence find the exact value of $\int_0^{\frac{\pi}{4}} 4\sin^3(\theta)d\theta$.

8 Using suitable expansions, find the value of $\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \cos^4(x)dx$.

Reasoning and communication

9 Prove by induction that $[\cos(\theta) + i\sin(\theta)]^n = \cos(n\theta) + i\sin(n\theta)$, $n \geq 1$, $n \in \mathbf{Z}$.

10 If $z = \cos(\theta) + i\sin(\theta)$, simplify

a $z - \dfrac{1}{z}$ b $z^2 - \dfrac{1}{z^2}$ c $z^n - \dfrac{1}{z^n}$

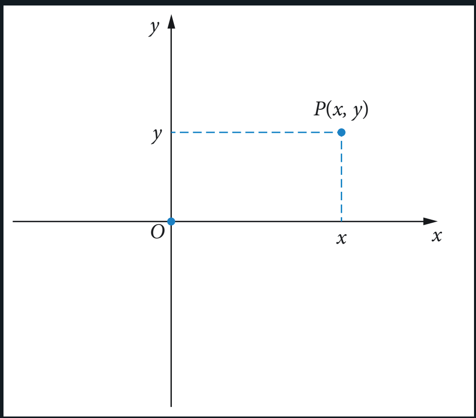
CHAPTER SUMMARY

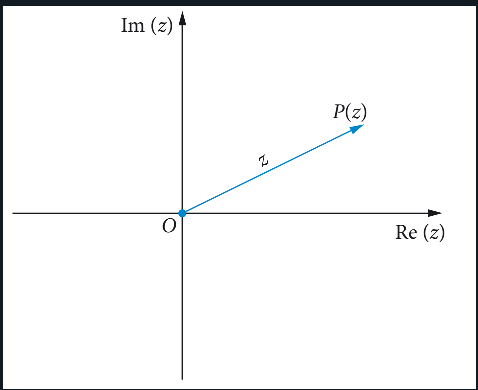
COMPLEX NUMBERS AND DE MOIVRE'S THEOREM

2

- The **imaginary number *i*** is the number such that $i = \sqrt{-1}$.
- A **complex number** is a number that can be written in the form $a + ib$, where a and b are real numbers.

  A complex number is often denoted by the letter z, so $z = a + ib$.
- For a complex number z, where $z = a + ib$ (where a and b are real numbers), the **complex conjugate** of z is denoted by $\overline{z}$ and $\overline{z} = a - ib$.
- The complex number $z = x + yi$ (where $x, y \in \boldsymbol{R}$) can be represented geometrically on an Argand diagram as the point $P(x, y)$ or the vector **z** or **OP**.

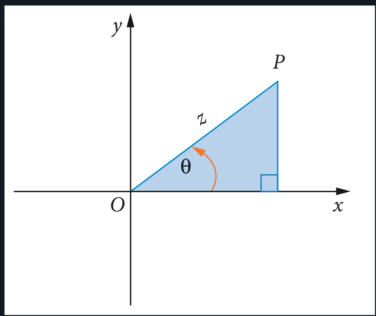




- For complex numbers $a + ib$ and $c + id$ (where a, b, c and d are real numbers), $a + ib = c + id$ if and only if $a = c$ AND $b = d$.
- To **realise the denominator** of a complex number, multiply the number by 1 in the form $\frac{\overline{z}}{\overline{z}}$.
- The **real part** of $z = a + ib$ is denoted by Re (z), where Re $(z) = a$, and the **imaginary part** of $z = a + ib$ is denoted by Im (z), where Im $(z) = b$.

  If Re $(z) = 0$, then z is *purely imaginary*.

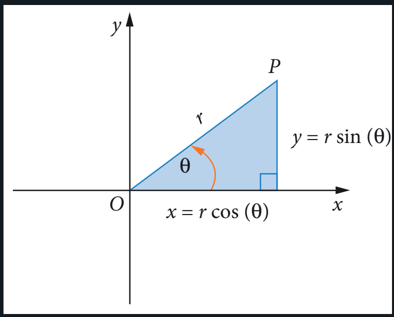
  If Im $(z) = 0$, then z is *purely real* or just *real*.
- Definition: the **Argument of *z***, or **arg** (z).



arg $(z) = \theta$

If a complex number is represented by a point P in the complex plane, then the **argument of *z***, denoted **arg** (z), is the angle θ that OP makes with the positive real axis O_x, with the angle measured anticlockwise from O_x. The **principal value** of the argument is the one in the interval $(-\pi, \pi]$. The argument of 0 is not defined.

- Definition: the **Polar form of *z***



Let $\arg(z) = \theta$ and $|z| = r$.

If z is a **non-zero** complex number, then $\boldsymbol{z = r[\cos(\theta) + i\sin(\theta)]}$ is the **polar** form of z.

- $\boldsymbol{z = r[\cos(\theta) + i\sin(\theta)]}$ is also written as $\boldsymbol{z = r\,\text{cis}(\theta)}$

- If $z_1 = r_1[\cos(\theta_1) + i\sin(\theta_1)]$ and $z_2 = r_2[\cos(\theta_2) + i\sin(\theta_2)]$ are two complex numbers, then
  - their **product** is $z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$
  - $\arg(z_1z_2) = \arg(z_1) + \arg(z_2)$ and $\text{mod}(z_1z_2) = \text{mod}(z_1) \times \text{mod}(z_2)$

- If $z_1 = r_1[\cos(\theta_1) + i\sin(\theta_1)]$ and $z_2 = r_2[\cos(\theta_2) + i\sin(\theta_2)]$ are two complex numbers, then $\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]$.

  If $z = r[\cos(\theta) + i\sin(\theta)]$ is a complex number, then $z^{-1} = \frac{1}{z} = \frac{1}{r}\left[\cos(-\theta) + i\sin(-\theta)\right]$.

  OR

  If $z_1 = r_1[\cos(\theta_1) + i\sin(\theta_1)]$ and $z_2 = r_2[\cos(\theta_2) + i\sin(\theta_2)]$ are two complex numbers, then $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$ and

  $\text{mod}\left(\frac{z_1}{z_2}\right) = \frac{\text{mod}(z_1)}{\text{mod}(z_2)}$.

- If $z = r[\cos(\theta) + i\sin(\theta)]$ is a complex number, then $\arg(z^{-1}) = -\arg(z)$ and $\text{mod}(z^{-1}) = \frac{1}{\text{mod}(z)}$.

- If $z_1 = r_1[\cos(\theta_1) + i\sin(\theta_1)]$, $z_2 = r_2[\cos(\theta_2) + i\sin(\theta_2)], \ldots,$ $z_n = r_n[\cos(\theta_n) + i\sin(\theta_n)]$ are multiple complex numbers, then

  $$z_1z_2 \ldots z_n = r_1r_2 \ldots r_n[\cos(\theta_1 + \theta_2 + \ldots + \theta_n) + i\sin(\theta_1 + \theta_2 + \ldots + \theta_n)].$$

  OR

  $\arg(z_1z_2 \ldots z_n) = \arg(z_1) + \arg(z_2) + \ldots + \arg(z_n)$ and

  $|z_1z_2 \ldots z_n| = |z_1||z_2| \ldots |z_n|$.

- $\arg(z^n) = n \times \arg(z)$ and $|z^n| = |z|^n$

  and

  $\arg(z^{-n}) = -n \times \arg(z)$ and $|z^{-n}| = |z|^{-n}$.

- **De Moivre's theorem**

  If $z = \cos(\theta) + i\sin(\theta)$ is a complex number, then $z^n = \cos(n\theta) + i\sin(n\theta)$, $\forall\ n \in \boldsymbol{Z}$.

- De Moivre's theorem is useful for deriving or proving a number of trigonometric identities.

- Expansions using De Moivre's theorem and the binomial theorem can also be used to integrate powers of trigonometric functions.

CHAPTER REVIEW

COMPLEX NUMBERS AND DE MOIVRE'S THEOREM

2

Multiple choice

1 Example 1 The roots of $x^2 - 2x + 10 = 0$ are

A $-1 \pm 3i$ B $1 \pm i\sqrt{3}$ C $1 \pm 3i$

D $-1 \pm i\sqrt{3}$ E none of the above

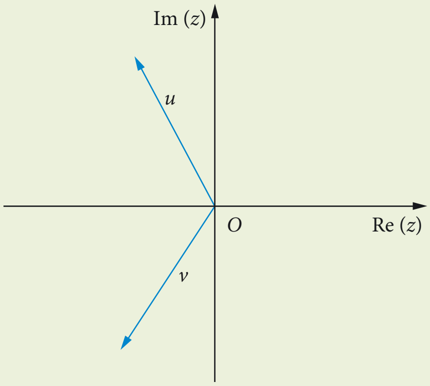
2 Example 1 The value of i^{74} is:

A 1 B -1 C i D $-i$ E 74

3 Example 2 The value of $\overline{3 - 2i + i(5 + 4i)}$ is:

A $1 - 3i$ B $-1 + 3i$ C $-1 - 5i$ D $-1 - 3i$ E $7 - 3i$

4 Example 3 Consider the vectors **u** and **v** representing the complex numbers u and v respectively, as shown in the diagram.



Which of the following statements is true?

A $u = v$ B $u = -v$ C $-u = -v$ D $\bar{u} = \bar{v}$ E $u = \bar{v}$

5 Example 11 The principal value of the argument of $2\text{cis}\left(\frac{31\pi}{6}\right)$ is:

A $-\frac{5\pi}{6}$ B $\frac{-\pi}{6}$ C $\frac{7\pi}{6}$ D $\frac{\pi}{6}$ E $\frac{5\pi}{6}$

Short answer

6 Example 4 Simplify $(3 - 2i)(1 + 4i) - (2 + 5i)^2$.

7 Example 5 Realise the denominator of $\frac{5+i}{2-i}$.

8 Example 6 Express $\left(2-i\sqrt{3}\right)^{-2}$ in the form $a + bi$. Hence find:

a $\text{Re}\left[\left(2-i\sqrt{3}\right)^{-2}\right]$

b $\text{Im}\left[\left(2-i\sqrt{3}\right)^{-2}\right]$

9 Example 7 Write down the complex number z in mod-arg form given that $\arg(z)=\frac{8\pi}{17}$ and $\text{mod}(z)=2\sqrt{5}$.

10 Example 8 Express $\sqrt{3}-i$ in mod-arg form.

11 Example 9 Express $4\left[\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)\right]$ in Cartesian form.

12 Example 14 If $z=2\left[\cos\left(\frac{3\pi}{4}\right)+i\sin\left(\frac{3\pi}{4}\right)\right]$, $w=3\left[\cos\left(\frac{5\pi}{9}\right)+i\sin\left(\frac{5\pi}{9}\right)\right]$ and $u=5\left[\cos\left(\frac{\pi}{12}\right)+i\sin\left(\frac{\pi}{12}\right)\right]$, find

a $\arg(zwu)$

b $|zwu|$

Application

13 Evaluate $\left(\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)^9$, giving your answer in Cartesian form.

14 Write the expansions of

a $\cos(A \pm B)$

b $\sin(A \pm B)$.

Hence prove that

c $[\cos(\theta) + i\sin(\theta)][\cos(\lambda) + i\sin(\lambda)] = \cos(\theta + \lambda) + i\sin(\theta + \lambda)$

d $[\cos(\theta) - i\sin(\theta)][\cos(\lambda) - i\sin(\lambda)] = \cos(-\theta - \lambda) + i\sin(-\theta - \lambda)$

15 Use De Moivre's theorem and the binomial theorem to expand $[\cos(\theta) + i\sin(\theta)]^3$ in two ways. Hence, or otherwise, find the value of $\int_{-1}^{2}\cos^3(8\alpha)\,d\alpha$. Give your answer correct to 3 significant figures.

16 Evaluate $\left\{2\left[\cos\left(\frac{3\pi}{5}\right)-i\sin\left(\frac{3\pi}{5}\right)\right]\right\}^{15}$, giving your answer in Cartesian form.

17 Prove the identity $|z_1z_2| = |z_1||z_2|$ for any two complex numbers z_1, z_2. Hence prove by mathematical induction that $|z_1z_2z_3 \ldots z_n| = |z_1||z_2||z_3| \ldots |z_n|$ for all positive integers n.

18 Simplify each of the following.

a $\frac{1}{\cos(5x)+i\sin(5x)}$

b $\frac{3\left[\cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right)\right]}{4\left[\cos\left(\frac{\pi}{5}\right)+i\sin\left(\frac{\pi}{5}\right)\right]}$

c $\frac{9[\cos(\beta)+i\sin(\beta)]}{3[\cos(4\beta)+i\sin(4\beta)]}$

19 Express $-\sqrt{2}-i\sqrt{2}$ in mod-arg form. Hence find the value of:

a $\arg\left(-\sqrt{2}-i\sqrt{2}\right)^{-1}$

b $\text{mod}\left(-\sqrt{2}-i\sqrt{2}\right)^{-1}$

9780170250306

20 Find the value of each of the following in mod-arg form.

a $\dfrac{1}{\left(4-4i\sqrt{3}\right)\left(-\sqrt{3}+i\right)}$

b $\left[\left(-1-i\sqrt{3}\right)(2+2i)\right]^{-1}$

21 Simplify each of the following.

a $\dfrac{\left[\text{cis}\left(\frac{5\pi}{7}\right)\right]^{3}\left[\text{cis}\left(\frac{6\pi}{7}\right)\right]^{5}}{\left[\text{cis}\left(\frac{2\pi}{7}\right)\right]^{6}}$

b $\dfrac{\left[\cos(4\alpha)+i\sin(4\alpha)\right]^{7}}{\left[\cos(9\alpha)+i\sin(9\alpha)\right]^{-4}\left[\cos(9\alpha)-i\sin(9\alpha)\right]^{3}}$

22 By expanding $[\cos(\theta) + i\sin(\theta)]^2$ in two different ways, derive expressions for $\cos(2\theta)$ and $\sin(2\theta)$. Hence show that:

a $\tan(2\alpha)=\dfrac{2\tan(\alpha)}{1-\tan^{2}(\alpha)}$

b $\tan(15°)=2-\sqrt{3}$

Practice quiz